# On the Maximum Number of Edges in a Hypergraph with a Unique Perfect Matching

Deepak Bal* Andrzej Dudek† Zelealem B. Yilma*

**Abstract**

In this note, we determine the maximum number of edges of a $k$-uniform hypergraph, $k \geq 3$, with a unique perfect matching. This settles a conjecture proposed by Snevily.

## 1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$, $\mathcal{E} \subseteq \binom{V}{k}$, be a $k$-uniform hypergraph (or $k$-graph) on $km$ vertices for $m \in \mathbb{N}$. A perfect matching in $\mathcal{H}$ is a collection of edges $\{M_1, M_2, \ldots, M_m\} \subseteq \mathcal{E}$ such that $M_i \cap M_j = \emptyset$ for all $i \neq j$ and $\bigcup_i M_i = V$. In this note we are interested in the maximum number of edges of a hypergraph $\mathcal{H}$ with a unique perfect matching. Hetyei observed (see, *e.g.*, [1, 2, 3]) that for ordinary graphs (*i.e.* $k = 2$), this number cannot exceed $m^2$. To see this, note that at most two edges may join any pair of edges from the matching. Thus the number of edges is bounded from above by $m + 2\binom{m}{2} = m^2$. Hetyei also provides a unique graph satisfying the above conditions. His construction can be easily generalized to uniform hypergraphs (see Section 2 for details). Snevily [4] anticipated that such generalization is optimal. Here we present our main result.

**Theorem 1.1.** *For integers $k \geq 2$ and $m \geq 1$ let*

$$f(k, m) = m + b_{k,2}\binom{m}{2} + b_{k,3}\binom{m}{3} + \cdots + b_{k,k}\binom{m}{k},$$

*where*

$$b_{k,\ell} = \frac{\ell - 1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell - i)}{k}.$$

*Let $\mathcal{H} = (V, \mathcal{E})$ be a $k$-graph of order $km$ with a unique perfect matching. Then*

$$|\mathcal{E}| \leq f(k, m). \tag{1.1}$$

*Moreover,* (1.1) *is tight.*

In particular, if $\mathcal{H} = (V, \mathcal{E})$ is a 3-uniform hypergraph of order $3m$ with a unique perfect matching, then

$$|\mathcal{E}| \leq f(3, m) = m + 9\binom{m}{2} + 18\binom{m}{3} = \frac{5m}{2} - \frac{9m^2}{2} + 3m^3.$$

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
†Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA

## 2 Construction

In this section, we provide a recursive construction of a hypergraph $\mathcal{H}_m^*$ of order $km$ with a unique perfect matching and containing exactly $f(k,m)$ edges.

Let $\mathcal{H}_1^*$ be a $k$-graph on $k$ vertices with exactly one edge. Trivially, this graph has a unique perfect matching. Suppose we already constructed a $k$-graph $\mathcal{H}_{m-1}^*$ on $k(m-1)$ vertices with a unique perfect matching. To construct the graph $\mathcal{H}_m^*$ on $km$ vertices, add $k-1$ new vertices to $\mathcal{H}_{m-1}^*$ and add all edges containing at least one of these new vertices. Then, add another new vertex and draw the edge containing the $k$ new vertices. Formally, let

$$M_i = \{k(i-1)+1, \dots, ki\} \text{ for } i = 1, \dots, m. \tag{2.1}$$

Let $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$ , $m \geq 1$, be a $k$-graph on $km$ vertices with the vertex set

$$V_m = \{1, \dots, km\} = \bigcup_{i=1}^{m} M_i$$

and the edge set (defined recursively)

$$\mathcal{E}_m = \mathcal{E}_{m-1} \cup \left\{ E \in \binom{V_m}{k} : E \cap M_m \neq \emptyset,\ km \notin E \right\} \cup \{M_m\},$$

where $\mathcal{E}_0 = \emptyset$.

Note that $\mathcal{H}_m^*$ has a unique perfect matching, namely, $\mathcal{M}_m = \{M_1, M_2, \dots, M_m\}$. To see this, observe that the vertex $km$ is only included in edge $M_m$. Hence, any matching must include $M_m$. Removing all vertices in $M_m$, we see that $M_{m-1}$ must be also included and so on. We call the elements of $\mathcal{M}_m$, *matching edges*.

**Claim 2.1.** *The $k$-graph $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$ satisfies $|\mathcal{E}_m| = f(k,m)$.*

*Proof.* For $\ell = 1, 2, \dots, k$, let $\mathcal{B}_\ell$ be the set of edges that intersect exactly $\ell$ matching edges, *i.e.*,

$$\mathcal{B}_\ell = \left\{ E \in \mathcal{E}_m : \sum_{i=1}^{m} \mathbf{1}_{E \cap M_i \neq \emptyset} = \ell \right\}.$$

Note that $\mathcal{E}_m = \bigcup_\ell \mathcal{B}_\ell$. Clearly, $|\mathcal{B}_1| = |\{M_1, \dots, M_m\}| = m$, giving us the first term in $f(k,m)$. Now we show that $|\mathcal{B}_\ell| = b_{k,\ell}\binom{m}{\ell}$ for $\ell = 2, \dots, k$. Let $\mathcal{L} = \{M_{i_1}, M_{i_2} \dots, M_{i_\ell}\} \subseteq \mathcal{M}_m$ be any set of $\ell$ matching edges with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$. Let $\mathcal{G}$ be the collection of $k$-sets on the vertex set of $\mathcal{L}$ which intersect all of $M_{i_1}, \dots, M_{i_\ell}$. The principle of inclusion and exclusion (conditioning on the number of $k$-sets that do not intersect a given subset of matching edges) yields that

$$|\mathcal{G}| = \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell-i)}{k}.$$

Now note that due to the symmetry of the roles of the vertices in $\mathcal{G}$, each vertex belongs to the same number of edges of $\mathcal{G}$, say $\eta$. Consequently, the number of pairs $(x, E)$, $x \in E \in \mathcal{G}$ equals $k\ell\eta$.

On the other hand, since every edge of $\mathcal{G}$ consists of $k$ vertices we get that the number of pairs is equal to $|\mathcal{G}|\, k$, implying that $\eta = |\mathcal{G}|\,/\ell$.

By construction, $E \in \mathcal{G}$ implies $E \in \mathcal{B}_\ell$ unless vertex $ki_\ell$ is in $E$. As

$$|\{E \in \mathcal{G} : ki_\ell \in E\}| = \eta = |\mathcal{G}|\,/\ell,$$

the number of edges of $\mathcal{B}_\ell$ on the vertex set of $\mathcal{L}$ equals

$$\frac{\ell-1}{\ell}\,|\mathcal{G}| = b_{k,\ell}. \tag{2.2}$$

As this argument applies to any choice of $\ell$ matching edges, we have $|\mathcal{B}_\ell| = b_{k,\ell}\binom{m}{\ell}$, thus proving the claim. □

**Corollary 2.2.** *For all integers $k \geq 2$ and $m \geq 1$,*

$$f(k,m) = m + \sum_{i=1}^{m-1}\left[\binom{k(i+1)-1}{k} - \binom{ki}{k}\right].$$

*Proof.* We prove this by counting the edges of $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$ in a different way. Let $a_m = |\mathcal{E}_m|$, $m \geq 1$. Then it is easy to see that the following recurrence relation holds: $a_1 = 1$ and

$$a_m = a_{m-1} + \binom{km-1}{k} - \binom{k(m-1)}{k} + 1 \text{ for } m \geq 2, \tag{2.3}$$

where the first binomial coefficient counts all the edges that do not contain vertex $km$; the second coefficient counts all the edges which do not intersect the matching edge $M_m$ (*cf.* (2.1)); and the term 1 stands for $M_m$ itself. Summing (2.3) over $m, m-1, \ldots, 2$ gives the desired formula. □

Note that $\mathcal{H}_m^*$ proves that (1.1) is tight. However, in contrast to the case of $k = 2$, there are hypergraphs on $km$ vertices containing a unique perfect matching and $f(k,m)$ edges which are not isomorphic to $\mathcal{H}_m^*$. For example, if $m = 2$, consider an edge $E \in \mathcal{H}_2^*$, $E \neq M_1, M_2$. Let $\bar{E}$ be the complement of $E$, *i.e.*, $\bar{E} = \{1, \ldots, 2k\} \setminus E$. Then, the hypergraph obtained from $\mathcal{H}_2^*$ by replacing $E$ with $\bar{E}$ provides a non-isomorphic example for the tightness of (1.1).

## 3 Proof of Theorem 1.1

We start with some definitions. We use the terms "edge" and "$k$-set" interchangeably.

**Definition 3.1.** *Given any collection of $2 \leq \ell \leq k$ disjoint edges $\mathcal{L} = \{M_1, \ldots, M_\ell\}$, we call a collection of edges $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ a* covering *of $\mathcal{L}$ if*

- *$C_i \cap M_j \neq \emptyset$ for all $i, j \in \{1, \ldots, \ell\}$, and*
- *$\bigcup_i C_i = \bigcup_i M_i$.*

Note that the second condition forces the edges in a covering to be disjoint.

**Definition 3.2.** *Let $\mathcal{L}$ be as in Definition 3.1, let $\mathcal{C}$ be a covering of $\mathcal{L}$ and let $C \in \mathcal{C}$. We say $C$* is of type $\vec{\boldsymbol{a}}$ *if*

- $\vec{\boldsymbol{a}} = (a_1, \ldots, a_\ell) \in \mathbb{N}^\ell$, $\sum_i a_i = k$ *and* $a_1 \geq a_2 \geq \ldots \geq a_\ell \geq 1$, *and*
- *there exists a permutation $\sigma$ of $\{1, 2, \ldots, \ell\}$ such that $\left|C \cap M_{\sigma(i)}\right| = a_i$ for each $1 \leq i \leq \ell$.*

*Let $\mathcal{A}_{k,\ell} = \{\vec{\boldsymbol{a}} = (a_1, \ldots, a_\ell) \in \mathbb{N}^\ell : a_1 \geq a_2 \geq \ldots \geq a_\ell \geq 1$ and $a_1 + \ldots + a_\ell = k\}$.*

Given a vector $\vec{\boldsymbol{a}} \in \mathcal{A}_{k,\ell}$, let $\mathcal{C}_{\vec{\boldsymbol{a}}}$ be the collection of all coverings $\mathcal{C}$ of $\mathcal{L}$ such that every $C \in \mathcal{C}$ is of type $\vec{\boldsymbol{a}}$. In other words, $\mathcal{C}_{\vec{\boldsymbol{a}}}$ consists of coverings using only edges of type $\vec{\boldsymbol{a}}$. We claim that $\mathcal{C}_{\vec{\boldsymbol{a}}}$ is not empty for every $\vec{\boldsymbol{a}} \in \mathcal{A}_{k,\ell}$. Indeed, for $i = 0, \ldots, \ell - 1$ let $\sigma_i$ be a permutation of $\{1, 2, \ldots, \ell\}$ (clockwise rotation) obtained by a cyclic shift by $i$, *i.e.*, $\sigma_i(j) = j + i \pmod{\ell}$. We form $C_i$ by picking $a_{\sigma_i(j)}$ items from $M_j$ for each $1 \leq j \leq \ell$. As $\sum_i a_{\sigma_i(j)} = k$, we may pick the $\ell$ edges $C_i$ to be disjoint, thereby obtaining a covering.

*Proof of Theorem 1.1.* Let $\mathcal{H} = (V, \mathcal{E})$ be a $k$-graph of order $km$ with the unique perfect matching $\mathcal{M} = \{M_1, \ldots, M_m\}$. We show that $|\mathcal{E}| \leq f(k, m)$.

We partition the edges into collections of edges which intersect exactly $\ell$ of the matching edges. That is, for $\ell = 1, \ldots, k$, we set

$$\mathcal{B}_\ell = \left\{ E \in \mathcal{E} : \sum_{i=1}^{m} \mathbf{1}_{E \cap M_i \neq \emptyset} = \ell \right\}.$$

Clearly, $|\mathcal{E}| = \sum_{\ell=1}^{k} |\mathcal{B}_\ell|$. Once again, $|\mathcal{B}_1| = m$. We will show, by contradiction, that $|\mathcal{B}_\ell| \leq b_{k,\ell}\binom{m}{\ell}$ for all $2 \leq \ell \leq k$.

Suppose that $|\mathcal{B}_\ell| > b_{k,\ell}\binom{m}{\ell}$ for some $2 \leq \ell \leq k$. Then, by the pigeonhole principle, there exists some set of $\ell$ matching edges, say, without loss of generality, $\mathcal{L} = \{M_1, \ldots, M_\ell\}$ such that

$$|\mathcal{B}_\ell \cap \mathcal{H}[\mathcal{L}]| \geq b_{k,\ell} + 1, \tag{3.1}$$

where $\mathcal{H}[\mathcal{L}]$ denotes the sub-hypergraph of $\mathcal{H}$ spanned by the vertices in $\bigcup_{i=1}^{\ell} M_i$. Let $\mathcal{G}$ be the collection of all $k$-sets on $\bigcup_i M_i$ that intersect every $M_i \in \mathcal{L}$. That is

$$\mathcal{G} = \left\{ A : |A| = k, A \cap M_i \neq \emptyset \text{ for each } 1 \leq i \leq \ell \text{ and } A \subseteq \bigcup_i M_i \right\}.$$

As in (2.2), we have

$$b_{k,\ell} = \frac{\ell - 1}{\ell} |\mathcal{G}| = \frac{\ell - 1}{\ell} \sum_{\vec{\boldsymbol{a}} \in \mathcal{A}_{k,\ell}} |\mathcal{G}_{\vec{\boldsymbol{a}}}|,$$

where $\mathcal{G}_{\vec{\boldsymbol{a}}}$ is the collection of $k$-sets of type $\vec{\boldsymbol{a}}$. Hence, by (3.1) we get

$$|\mathcal{B}_\ell \cap \mathcal{H}[\mathcal{L}]| \geq \frac{\ell - 1}{\ell} \sum_{\vec{\boldsymbol{a}} \in \mathcal{A}_{k,\ell}} |\mathcal{G}_{\vec{\boldsymbol{a}}}| + 1,$$

and consequently, there exists some type $\vec{\boldsymbol{a}}$ such that

$$|\mathcal{B}_\ell \cap \mathcal{G}_{\vec{\boldsymbol{a}}}| \geq \frac{\ell-1}{\ell}\, |\mathcal{G}_{\vec{\boldsymbol{a}}}| + 1. \tag{3.2}$$

Recall that $|\mathcal{C}| = \ell$ and that $\mathcal{C}_{\vec{\boldsymbol{a}}}$ is the nonempty collection of all coverings $\mathcal{C}$ of $\mathcal{L}$ such that every $C \in \mathcal{C}$ is of type $\vec{\boldsymbol{a}}$. By symmetry, every $k$-set $A \in \mathcal{G}_{\vec{\boldsymbol{a}}}$ belongs to exactly

$$\frac{|\mathcal{C}_{\vec{\boldsymbol{a}}}|\ell}{|\mathcal{G}_{\vec{\boldsymbol{a}}}|}$$

coverings $\mathcal{C} \in \mathcal{C}_{\vec{\boldsymbol{a}}}$. Since no $\mathcal{C} \in \mathcal{C}_{\vec{\boldsymbol{a}}}$ is contained in $\mathcal{H}$ (otherwise we could replace $\mathcal{L}$ by $\mathcal{C}$ to obtain a different perfect matching, contradicting the uniqueness of $\mathcal{M}$), the number of $k$-sets in $\mathcal{G}_{\vec{\boldsymbol{a}}}$ that are not in $\mathcal{B}_\ell$ is at least

$$|\mathcal{C}_{\vec{\boldsymbol{a}}}| \Bigg/ \frac{|\mathcal{C}_{\vec{\boldsymbol{a}}}|\ell}{|\mathcal{G}_{\vec{\boldsymbol{a}}}|} = \frac{|\mathcal{G}_{\vec{\boldsymbol{a}}}|}{\ell}.$$

That means,

$$|\mathcal{B}_\ell \cap \mathcal{G}_{\vec{\boldsymbol{a}}}| \leq \frac{\ell-1}{\ell}\, |\mathcal{G}_{\vec{\boldsymbol{a}}}|$$

which contradicts (3.2). Thus, $|\mathcal{B}_\ell| \leq b_{k,\ell}\binom{m}{\ell}$, as required. $\square$

# References


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